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LOCAL STRUCTURE OF GRAPHS WITH $\lambda=\mu=2$, $a_2=4$

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Several properties of graphs with $\lambda=\mu=2, a_2=4$ are studied. It is proved that such graphs are locally unions of triangles, hexagons or heptagons. As a consequence, a distance regular graph with intersection array $(13,10,7;1,2,7)$ does not exist.

1. Introduction

In this paper we study graphs Γ which satisfy the following axioms:

1. ($\lambda = 2$.) For any two adjacent vertices in Γ there are exactly two vertices adjacent to both.
2. ($\mu = 2$.) For any two vertices at distance 2, there are exactly two vertices adjacent to both.
3. ($a_2 = 4$.) For any pair of vertices $p, q \in \Gamma$ at distance 2, there are exactly 4 vertices adjacent to q and at distance 2 from p.

We use the following notations:

$x \sim y$	$x, y \in \Gamma$ are adjacent,
$d(x,y)$	distance between two vertices,
$\Gamma(x)$	set of vertices adjacent to x,
$\Gamma_2(x)$	set of vertices at distance 2 of x.

We shall also assume that Γ is connected. An immediate consequence of the axioms is the following theorem.

Theorem 1. *If Γ is connected, then Γ is regular, i.e., every vertex has the same degree k.*

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Proof. This is true for every amply regular graph with $\mu > 1$. (See for instance [1], prop. 1.1.2.) ∎

2. Examples

The following are examples of distance-regular graphs with $\lambda = \mu = 2$ and $a_2 = 4$. These examples are taken from [1].

1. The complete graph K_4 on 4 vertices satisfies the axioms in a trivial way.

2. There are two strongly regular graphs with parameters $v = 16, k = 6, \lambda = \mu = 2$ (and hence $a_2 = 4$). They are the 4×4 grid (i.e., the direct product $K_4 \times K_4$) and the Shrikhande graph. Note that the Shrikhande graph is locally a hexagon.

3. The direct products of a number of copies of the K_4 graph and/or the Shrikhande graph are called Doob graphs and also satisfy our axioms. These graphs are distance regular.

4. There is a unique distance regular graph with intersection array $(7,4,1;1,2,7)$ on 24 vertices. This graph also satisfies our axioms. It is an antipodal 3-cover of K_8 and can be constructed as follows: vertices are elements of $(\mathbb{Z}/7\mathbb{Z})^2 \setminus (0,0)$, where we identify (a,b) and $(-a,-b)$. Two vertices (a,b) and (c,d) are adjacent iff $ad - bc = \pm 1$. Note that this graph is locally a heptagon.

3. Local structure of Γ

In what follows we assume that Γ has only a finite number of vertices.

A connected induced subgraph of Γ of degree 2 and size n is called a *true n-gon*. Note that for any $x \in \Gamma$, the subgraph induced on $\Gamma(x)$ is always a disjoint union of true polygons.

Most proofs in this paper make use of the following important lemma:

Lemma 1. *Consider a true pentagon $p_0 p_1 \cdots p_4$ in Γ. Let q be the unique vertex $\neq p_0$ adjacent to both p_1 and p_4. If q is not adjacent to p_0 then q must be adjacent to both p_2 and p_3.*

Proof. Assume that $p_0 \not\sim q$.

Because $\lambda = 2$, there are 2 vertices a, b adjacent to q and p_1 and 2 vertices c, d adjacent to p_0 and p_1. Any vertex belonging to both $\{a,b\}$ and $\{c,d\}$ is adjacent to both p_0 and q. But also p_1 and p_4 are adjacent to p_1 and q, and hence, because $\mu = 2$, the sets $\{a,b\}$ and $\{c,d\}$ must be disjoint. As $p_0 \not\sim q$, also p_0 and q do not belong to $\{a,b,c,d\}$, and hence a, b, c, d, p_0, q are all different.

Now, $a, b, c, d, p_2 \in \Gamma(p_1) \cap \Gamma_2(p_4)$, and because $a_2 = 4$ this implies that p_2 is one of a, b, c, d. As $p_2 \not\sim p_0$ we must have $p_2 \sim q$. Interchanging the role of p_1 and p_4 in the above argument proves that also p_3 is adjacent to q. ∎

Choose a vertex in Γ and denote it by the symbol ∞. Choose a vertex 0 adjacent to ∞. There is a unique true polygon in $\Gamma(\infty)$ that contains the vertex 0. Denote the size of this polygon by n, and denote the successive vertices of the polygon by successive elements $0,1,2,\ldots,n-1$ of $\mathbb{Z}/n\mathbb{Z}$. We shall use the notational convention $\bar{a}\stackrel{\text{def}}{=}-a \pmod n), \forall a$.

Consider any pair of vertices $x,y\in\mathbb{Z}/n\mathbb{Z}$. Apart from ∞ there is exactly one other vertex adjacent to both x and y. Denote this vertex by xy. Clearly $xy=yx$.

Now if x and y differ 2 $\pmod n$ then xy is adjacent to ∞ and denotes the same vertex as $\frac{1}{2}(x+y)$. Conversely, if xy, $x\neq y$ is adjacent to ∞ then $\lambda=2$ implies $xy\in\{x-1,x+1\}$ and $xy\in\{y-1,y+1\}$ and therefore $x-y=\pm 2$.

Also, because $\lambda=\mu=2$, if $xy=zu$ then the set $\Gamma(xy)\cap\Gamma(\infty)=\{x,y,z,u\}$ must have size 2, and hence $x=z, y=u$ or $x=u, y=z$.

The set of vertices constructed sofar consists of the vertex ∞, all vertices of the given n-gon in $\Gamma(\infty)$ and all vertices adjacent to exactly 2 non-adjacent vertices of that n-gon. We denote this set by $N(\infty,0)$. The above argument shows that $|N(\infty,0)|=(n^2-n+2)/2$.

The notational scheme used above depends on the choice of the vertex 0 adjacent to ∞ (which could have been chosen in n ways) and the choice of the vertex 1 adjacent to both 0 and ∞ (which could have been done in 2 ways). As a consequence, the proofs in this section remain valid when one of the transformations $x\mapsto\bar{x}$ or $x\mapsto x+a \pmod n$ for fixed a, is applied to all numbers involved. Whenever we use this argument in the proofs below, we will use the clause 'by symmetry'.

Lemma 2. *If $n>5$ then $01,03,23$ form a triangle.*

Proof. Consider the pentagon $0,1,2,3,03$. This is a true pentagon. Consider the vertex $x\neq 0$ which is adjacent to both 1 and 03. By Lemma 1 there are two possibilities: x is adjacent to 0, or x is adjacent to both 2 and 3. In the latter case, x must be either ∞ (which is however not adjacent to 03, as $n>5$) or 23 (which cannot be adjacent to 1). Hence x is adjacent to 0. As a consequence x is either ∞ or 01, and the first case is easily excluded because $n>5$ and therefore $03\not\sim\infty$. This proves that $01\sim 03$.

By symmetry also $12\sim 14$.

Consider the pentagon $1,2,3,03,01$. Again this is a true pentagon. Now consider the vertex $y\neq 1$ adjacent to both 2 and 01. If $y\sim 1$ then $y=12$ and then three vertices $2,14,01$ are adjacent to both 1 and 12, contradicting $\mu=2$. Hence $y\not\sim 1$ and then by Lemma 1, $y\sim 3$ and $y\sim 03$. Because also $y\sim 2$ we must have $y=\infty$ or $y=23$. The first case is easily excluded. ∎

Lemma 3. *If $n>5$, then the vertices $\infty,1,01,03,04,\ldots,0\bar{4},0\bar{3},0\bar{1},\bar{1},\infty$ form an n-gon in $\Gamma(0)$.*

Proof. Lemma 2 proves that $01\sim 03$ and by symmetry that $0\bar{1}\sim 0\bar{3}$.

Now take any $k\in\mathbb{Z}/n\mathbb{Z}, k\neq\bar{3},\bar{2},\bar{1},0,1,2$ and consider the pentagon $0,0k,k,k+1,0(k+1)$. If $0k$ and $0(k+1)$ are not adjacent, then this is a true pentagon. In this case Lemma 1 implies that ∞ is adjacent to either $0k$ or $0(k+1)$, a contradiction. Hence $0k$ is adjacent to $0(k+1)$. This proves the lemma. ∎

Lemma 4. *The size n of a polygon in $\Gamma(\infty)$ is either 3, 5, 6 or 7.*

Proof. Clearly $n \neq 4$, for otherwise $\infty, 1$ and 3 would be adjacent to both 0 and 2. It remains to be proved that $n < 8$. Assume $n > 6$.

In Lemma 3 it was proved that $0\bar{4} \sim 0\bar{3}$. By symmetry this proves that $(0+4)(\bar{4}+4) \sim (0+4)(\bar{3}+4)$, i.e., $04 \sim 14$. As a consequence 14 is adjacent to both 1 and 04 and therefore plays the same role in $N(0,1)$ as did 03 in $N(\infty,0)$. In a similar way $1\bar{2}$ in $N(0,1)$ corresponds to 01 in $N(\infty,0)$ (as $10 \sim 1\bar{2}$ by Lemma 2).

Applying Lemma 2 to $N(0,1)$ shows that $1\bar{2} \sim 14$. Also, applying Lemma 3 to $N(1,2)$ yields $12 \sim 14 \sim 15$. Hence, the vertices $1\bar{2}, 12, 15$ are adjacent to both 1 and 14 and therefore (as $\mu = 2$) $1\bar{2}$ must be equal to either 12 or 15. This implies $n = 7$. ∎

4. The case $n > 3$

Theorem 2. *If Γ contains a hexagon each vertex of which is adjacent to the same vertex $\infty \in \Gamma$, then this hexagon belongs to an induced subgraph isomorphic to the Shrikhande graph.*

Proof. The lemmas of the previous section prove that the subgraph induced on $N(\infty,0)$ has (at least) the following adjacencies (we only list adjacencies for typical vertices, others can be obtained by symmetry):

vertex	adjacent to					
∞	0	1	2	3	4	5
0	∞	1	01	03	05	5
01	0	1	14	45	23	03
03	0	01	23	3	34	05

Define a graph Δ with 24 vertices denoted by $\infty, 0, \ldots, 5, 01, \ldots, 45$ and with adjacencies as in the table above. The subgraph induced on $N(\infty,0)$ always contains an isomorphic copy of Δ. In particular, the Shrikhande graph, which satisfies the conditions of this theorem, must contain a copy of Δ. Now both Δ and the Shrikandhe graph are graphs with 16 vertices and degree 6. Hence Δ is isomorphic to the Shrikhande graph.

As both Γ and Δ satisfy $\lambda = \mu = 2$, no further adjacencies can occur in $N(\infty,0)$. ∎

A similar theorem can be proved for $n = 7$. We need some lemmas:

Lemma 5. *If $n = 7$, then 01, 23, 45, 06, 12, 34, 56 is a true heptagon and there is a vertex ∞' to which each of its vertices are adjacent.*

Proof. By Lemma 2 we know that $01 \sim 23$, and by symmetry, the given 7 vertices determine a heptagon.

Now consider the pentagon 01, 23, 3, 34, 56. We first prove that this is a 'true' pentagon. The vertices 03 and 2 are adjacent to the pair 3, 23, hence $23 \not\sim 34$ (for $\lambda=2$). Both 23 and 03 are adjacent to the pair 01, 3. Hence $34 \not\sim 01$. Both 34 and 36 are adjacent to the pair 3, 56, hence $23 \not\sim 56$.

Define ∞' to be the vertex $\neq 3$ adjacent to both 23 and 34. The two vertices adjacent to the pair 3, 23 are 2 and 03 and the two vertices adjacent to the pair 3, 34 are 4 and 36. Hence no vertex is adjacent to 23, 3 and 34 and in particular $\infty' \not\sim 3$. By Lemma 1, ∞' is adjacent to 01, 23, 34 and 56.

By symmetry, there is a vertex ∞'_1 adjacent to 23, 45, 56 and 01, and it is easily seen that this implies $\infty' = \infty'_1$. Proceeding in this way we may prove that ∞' is adjacent to every vertex of the given heptagon. As a consequence, the heptagon is 'true'. ∎

Lemma 6. *If $n=7$, then 03, 04, 14, 15, 25, 26, 36 is a true heptagon and there exists a vertex ∞'' to which each of its vertices are adjacent.*

Proof. The proof is similar as in the previous lemma, using the pentagon 01, 03, 04, 14, 15 with ∞'' adjacent to 03 and 15. We leave the rest of the proof to the reader. ∎

Theorem 3. *If Γ contains a heptagon each vertex of which is adjacent to the same vertex $\infty \in \Gamma$, then this heptagon is contained in an induced subgraph of size 24 which is distance regular with parameters $(7,4,1;1,2,7)$.*

Proof. The lemmas of this section and the previous prove that the subgraph induced on $N(\infty) \cup \{\infty', \infty''\}$ has (at least) the following adjacencies (we only list adjacencies for typical vertices, other adjacencies can be obtained by symmetry):

vertex	adjacent to						
∞	0	1	2	3	4	5	6
∞'	01	23	45	06	12	34	56
∞''	03	04	14	15	25	26	36
0	∞	1	01	03	04	06	6
01	0	1	15	56	∞'	23	03
03	0	01	23	3	36	∞''	04

We proceed in a similar way as in Theorem 2. Consider the graph Δ with vertices denoted by $\infty, \infty', \infty'', 0, \ldots, 6, 01, \ldots, 56$ and with adjacencies defined as in the table above. The subgraph induced on $N(\infty, 0)$ always contains an isomorphic copy of Δ.

In particular, the unique distance regular graph Γ' with parameters $(7,4,1;1,2,7)$ satisfies the conditions of this theorem, and therefore contains a copy of Δ. Both graphs Γ' and Δ have 24 vertices and degree 7. Hence they must be isomorphic. It also easily follows that no further adjacencies between the 24 vertices in $N(\infty,0)$ exist. ∎

As a consequence of Theorems 2 and 3 we may prove that the case $n=5$ cannot occur:

Theorem 4. *Γ cannot contain a pentagon in which each vertex is adjacent to the same vertex ∞.*

Proof. As before we number the vertices of the pentagon as $0,1,2,3,4$. Now $\Gamma(0)$ contains the following path of length 4: $04,4,\infty,1,01$. We shall first prove that $01 \not\sim 04$.

Assume the contrary. Define $S_1 \stackrel{\text{def}}{=} \Gamma(1)\cap\Gamma_2(4)$. Note that $3,04\in S_1$. Because $d(1,4)=2$ we have $|S_1|=4$. Denote the remaining two vertices by x_1,y_1. In a similar way, define $S_4 \stackrel{\text{def}}{=} \Gamma(4)\cap\Gamma_2(1)$. Note that $2,01\in S_4$ and denote the remaining two vertices of S_4 by x_4,y_4.

Now, because $\mu=2$, for every vertex $q\in S_1$ there are two adjacent vertices in $S_4\cup\{0,\infty\}$ and conversely for every vertex $q\in S_4$ there are two adjacent vertices in $S_1\cup\{0,\infty\}$. With the assumption above, this is only possible when both x_1 and y_1 are joined to both x_4 and y_4. But because also $x_4\sim 4\sim y_4$, this contradicts $\mu=2$.

Hence $01\not\sim 04$. As a consequence the path $04,4,\infty,1,01$ in $\Gamma(0)$ is part of an n-gon with $n>5$. By Lemma 4 and Theorems 2 and 3 it follows that $N(0,\infty)$ must be part of either a Shrikande graph, or a distance regular graph with parameters $(7,4,1;1,2,7)$. But then the subgraph induced on $\infty,0,\dots,4$ must be contained in such a graph, which is a contradiction, for neither graph contains such a pentagon. ∎

As an immediate consequence we may strengthen Lemma 4 to the following theorem:

Theorem 5. *Let ∞ be a given vertex in a graph Γ with parameters $\lambda=\mu=2, a_2=4$, then $\Gamma(\infty)$ is a disjoint union of triangles, hexagons and/or heptagons.* ∎

5. Application

Theorem 6. *There is no distance regular graph Γ with parameters* $(13,10,7;1,2,7)$.

Proof. The parameters of this graph imply $\lambda=\mu=2, a_2=4$.

Consider any vertex $\infty\in\Gamma$. By Theorem 5, $\Gamma(\infty)$ is a union of triangles, hexagons and/or heptagons. As $|\Gamma(\infty)|=13$ is not divisible by 3, there must be at least one heptagon. By Theorem 3, Γ must have an induced subgraph Δ with parameters $(7,4,1;1,2,7)$.

For any vertex $x\in\Delta$ define N_x to be the set of vertices of $\Gamma-\Delta$ adjacent to x. We have $|N_x|=13-7=6$. Consider two different vertices $x,y\in\Delta$. We shall prove that N_x and N_y are disjoint.

Assume $z\in N_x\cap N_y$. If $d(x,y)\leq 2$ then there are 2 vertices adjacent to both x and y in Δ plus at least one extra vertex z outside. This contradicts $\lambda=\mu=2$. Now, assume the distance between x and y is 3 in Δ, then consider $u\in\Gamma(y)\cap\Gamma_2(x)$. There are 4 vertices in Δ belonging to $\Gamma(u)\cap\Gamma_2(x)$. But in Γ also z belongs to $\Gamma(u)\cap\Gamma_2(x)$. This contradicts $a_2=4$.

Hence, all N_x are disjoint. This implies that Γ must contain at least $|\Delta|+6|\Delta|=24\cdot 7=168$ vertices. But $|\Gamma|=144$. Hence Γ does not exist. ∎

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References

[1] A. E. Brouwer, A. M. Cohen, and A. Neumaier: *Distance-regular Graphs*, Springer Verlag, Berlin, 1989.

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